TRANSPORT PHENOMENA IN A NONEQUILIBRIUM, PARTIALLY IONIZED GAS IN A MAGNETIC FIELD

Jean-Pierre Petit and Michel Larini   UDC 533.932

A small-parameter method in which the gas and electron temperatures can be different is used to solve the Boltzmann equation. The zeroth-approximation solutions are Maxwellian with different temperatures T_e and T_s. Transition to the BGK formalism on the basis of an extremely crude estimate of the frequency of electron collisions leads to numerical results which agree well with the available data. Then an extension of the Eucken method leads to analytic expressions for the nonequilibrium quasi-Lorentz transport coefficients.

The Relaxation-Time Problem. Transport phenomena in ionized gases are described by finding a common solution of the Boltzmann equations for each component of a mixture of electrons, ions, and neutral atoms. In each of these equations an integral term incorporates collisions between particles of the same species while the other terms are not related to collisions. For a nonequilibrium state the velocity distributions are assumed non-Maxwellian; the components are assumed to have different temperatures; the macroscopic velocities are assumed different; and gradients are assumed to exist.

An ionized gas in any nonequilibrium state tends toward equilibrium; the process can involve several steps with different time scales. Grad [1] suggested that the lighter components reach equilibrium first; then the heavier components reach equilibrium; and then all components reach equilibrium with each other as a result of collisions.

This suggestion was verified by Morse [2], who studied relaxation in an inhomogeneous binary mixture without gradients, with a small diffusion Mach number. Morse introduced the self-relaxation times τ_{ee} and τ_{ss} and calculated the characteristic times for exchange of kinetic energy (these times are denoted by a subscript "E") and for momentum exchange ("M"): $\tau_{E(e)}$, $\tau_{M(e)}$, $\tau_{E(s)}$, and $\tau_{M(s)}$. These times are related. Assuming a Coulomb interaction and using $\alpha = n_e/n_s$, we find

$$\frac{\tau_{ee}}{\tau_{ss}} \propto \frac{\varepsilon}{\alpha}, \quad \frac{\tau_{ee}}{\tau_{M(e)}} \propto \frac{1}{\alpha}, \quad \frac{\tau_{ee}}{\tau_{M(s)}} \propto \varepsilon^2, \quad \frac{\tau_{ee}}{\tau_{E(e)}} \propto \frac{\varepsilon^2}{\alpha}, \quad \frac{\tau_{ee}}{\tau_{E(s)}} \propto \varepsilon^2. \tag{1}$$

Johnson [3] recently proposed a simplification of the Boltzmann equation based on the introduction of the parameter $\varepsilon = (m_e/m_s)^{1/2}$; this approach leads to the equations

$$\frac{\partial f_e}{\partial t} + \vec{W}_e \cdot \frac{\partial f_e}{\partial \vec{r}} + \varepsilon^{-2} T_{ee} + T_{es} = 0,$$

$$\frac{\partial f_s}{\partial t} + \vec{W}_s \cdot \frac{\partial f_s}{\partial \vec{r}} + T_{se} + \varepsilon^{-1} T_{ss} = 0. \tag{2}$$

Here the term describing collisions between different species is governed in order of magnitude by the difference between the temperatures of the relaxation processes, as in [2]. These equations are valid for describing thermal processes, but if diffusion must be taken into account they must be modified. In general, the term T_{es}, corresponding to collisions between different species, has two relaxation times $\tau_{E(e)}$ and $\tau_{M(e)}$, which are quite different in magnitude [see (1)]. Our purpose below is to study transport phenomena in a nonequilibrium, partially ionized gas in magnetic and electric fields.

I. The Small-Parameter Method. To illustrate the use of this method we consider a binary mixture; in this case we have the two equations

$$\frac{1}{n_e}\left[\frac{\partial f_e}{\partial t}+\vec{c}_e\cdot\frac{\partial f_e}{\partial r}+\vec{V}\cdot\frac{\partial f_e}{\partial r}-\frac{e}{m_e}(\vec{E}+\vec{V}\times\vec{B}+\vec{c}_e\times\vec{B})\cdot\frac{\partial f_e}{\partial c_e}+T_{ee}+T_{es}\right]=0,\tag{3}$$

$$\frac{1}{n_s}\left[\frac{\partial f_s}{\partial t}+\vec{c}_s\cdot\frac{\partial f_s}{\partial r}+\vec{V}\cdot\frac{\partial f_s}{\partial r}+\frac{e}{m_s}(\vec{E}+\vec{V}\times\vec{B}+\vec{c}_s\times\vec{B})\cdot\frac{\partial f_s}{\partial c_s}+T_{ss}+T_{se}\right]=0.\tag{4}$$

We expand the unknown distribution functions:

$$f_e=n_e(F_e^0+\theta F_e^1+\theta^2 F_e^2+\ldots),\tag{5}$$

$$f_s=n_s(F_s^0+\theta F_s^1+\theta^2 F_s^2+\ldots),\tag{6}$$

where θ is the small parameter, defined below. We impose the conditions

$$\int f_e^0\,d_3W_e=\int f_e d_3W_e=n_e,\tag{7}$$

where

$$(n_e m_e+n_s m_s)\vec{V}=n_e m_e\vec{V}_e+n_s m_s\vec{V}_s,\tag{8}$$

$$\vec{V}_e=\frac{1}{n_e}\int\vec{W}_e f_e d_3W_e,\quad \vec{V}_s=\frac{1}{n_s}\int\vec{W}_s f_s d_3W_s,\tag{9}$$

$$\int\vec{W}_e f_e^0\,d_3W_e=n_e\vec{V},\tag{10}$$

$$\int\frac{1}{2}m_e(\vec{W}_e-\vec{V})^2 f_e^0\,d_3W_e=\int\frac{1}{2}m_e(\vec{W}_e-\vec{V})^2 f_e d_3W_e=\frac{3n_e kT_e}{2},\tag{11}$$

which lead to

$$\int f_e^1\,d_3W_e=0,\tag{12}$$

$$\int\vec{W}_e f_e^1\,d_3W_e=n_e(\vec{V}_e-\vec{V}),\tag{13}$$

$$\int\frac{1}{2}m_e(\vec{W}_e-\vec{V})^2 f_e^1\,d_3W_e=0.\tag{14}$$

Accordingly, the thermal relaxation processes are described by F_e^0 and F_s^0. It should be noted that the difference T_e-T_s is probably not simply a slight perturbation; on the other hand, the differences $(\vec{V}_e-\vec{V})/\langle c_e\rangle$ and $(\vec{V}_s-\vec{V})/\langle c_s\rangle$ are small (according to the assumption of a small diffusion Mach number).

All the terms in Eqs. (3) and (4) have characteristic times; for each component we have the relaxation times τ_{ee} and τ_{ss}, which govern energy and momentum exchange. For heavy components the times $\tau_{E(s)}$ and $\tau_{M(s)}$ are still equal, but for the electrons they are very different:

$$\tau_{M(e)}=\frac{m_e}{m_s}\tau_{E(e)}=\varepsilon^2\tau_{E(e)}.\tag{15}$$

Dimensional analysis for each of the variables leads to
$$t=\tau_{ref}t',\quad \vec{c}_e=\langle c_e\rangle\vec{\varphi}_e,\quad \vec{c}_s=\langle c_s\rangle\vec{\varphi}_s,\quad \vec{E}^*=\vec{E}+\vec{V}\times\vec{B},$$

$$\vec{E}=E\vec{\xi},\quad \vec{V}=V\vec{v},\quad \vec{B}=B\vec{\beta},\quad \vec{r}=L_{ref}\vec{p}.\tag{16}$$

II. The Time-Derivative Problem. The distribution functions are governed by the fields, the macroscopic properties, and their derivatives. We first consider the case in which there are only temperature gradients. Since the field continuously supplies energy to the gas, it cannot be in a steady state. The electron and gas temperatures increase, at rates described by the times $\tau_{E(e)}$ and $\tau_{E(s)}=\tau_{E(e)}/\alpha$. Since $\tau_{M(e)}\ll\tau_{E(e)}$, we can assume that $\vec{v}_e$ follows the temperature changes; i.e., over a time $t\ll\tau_{E(e)}$ the system is in a quasisteady state, so that we have

$$\frac{1}{n_e}\frac{\partial f_e}{\partial t}\simeq\frac{1}{\tau_{E(e)}}\left(\frac{\partial F_e^0}{\partial t'}+\theta\frac{\partial F_e^1}{\partial t'}+\ldots\right).\tag{17}$$

TABLE 1

	Electrons	Heavy component
~ 1	$$\alpha T_{ee}^0 - \beta_e (\vec{\varphi}_e \times \vec{\beta}) \cdot \frac{\partial \vec{F}_e^0}{\partial \varphi_e} = 0 \quad (27)$$	
~ ε	$$\alpha T_{ee}^1 + T_{es}^1 + \vec{\varphi}_e \cdot \frac{\partial \vec{F}_e^0}{\partial p} - \vec{\xi} \cdot \frac{\partial \vec{F}_e^0}{\partial \varphi_e} - \beta_e (\vec{\varphi}_e \times \vec{B}) \cdot \frac{\partial \vec{F}_e^{(1)}}{\partial \varphi_e} = 0 \quad (28)$$	$$T_{ss}^0 = 0 \quad (29)$$
~ ε²	$$\frac{\partial F_e^0}{\partial t'} + \vec{\varphi}_e \cdot \frac{\partial \vec{F}_e^1}{\partial p} + \vec{V} \cdot \frac{\partial \vec{F}_e^0}{\partial p} - \vec{\xi} \cdot \frac{\partial \vec{F}_e^1}{\partial \varphi_e} - \beta_e (\vec{\varphi}_e \times \vec{B}) \cdot \frac{\partial F_e^{(2)}}{\partial \varphi_e} + T_{es}^0 + \alpha T_{ee}^2 + T_{es}^2 = 0 \quad (30)$$	$$\frac{\partial F_s^0}{\partial t'} + \vec{\varphi}_s \cdot \frac{\partial \vec{F}_s^0}{\partial p} - \vec{\xi} \cdot \frac{\partial \vec{F}_s^0}{\partial \varphi_s} - \beta_e (\vec{\varphi}_s \times \vec{B}) \cdot \frac{\partial \vec{F}_s^0}{\partial \varphi_s} + T_{ss}^1 + \alpha T_{se}^0 = 0 \quad (31)$$

Multiplying the Boltzmann equation by the characteristic time τ, we find

$$\frac{\tau}{\tau_{E(e)}} \left(\frac{\partial F_e^0}{\partial t} + \theta \frac{\partial F_e^1}{\partial t} \right) + \frac{\langle c_e \rangle \tau}{L_e} \left[\vec{\varphi}_e \cdot \frac{\partial \vec{F}_e^0}{\partial p} + \theta \vec{\varphi}_e \cdot \frac{\partial \vec{F}_e^1}{\partial p} \right]$$

$$+ \frac{V\tau}{L_e} \left[\vec{V} \cdot \frac{\partial \vec{F}_e^0}{\partial p} + \theta \vec{V} \cdot \frac{\partial \vec{F}_e^1}{\partial p} \right] - \frac{cE^* \langle c_e \rangle \tau}{m_e \langle c_e \rangle^2} \left[\vec{\xi} \cdot \frac{\partial \vec{F}_e^0}{\partial \varphi_e} + \theta \vec{\xi} \cdot \frac{\partial \vec{F}_e^1}{\partial \varphi_e} \right]$$

$$- \frac{eB\tau}{m_e} (\vec{\varphi}_e \times \vec{B}) \left[\frac{\partial \vec{F}_e^0}{\partial \varphi_e} + \theta \frac{\partial \vec{F}_e^1}{\partial \varphi_e} + \theta^2 \frac{\partial \vec{F}_e^2}{\partial \varphi_e} \right]$$

$$+ \frac{\tau}{\tau_{ee}} T_{ee}^{(0)} + \theta \frac{\tau}{\tau_{ee}} T^{(1)} + \theta^2 \frac{\tau}{\tau_{ee}} T^{(2)}$$

$$+ \frac{\tau}{\tau_{E(e)}} T_{es}^{(0)} + \frac{\tau}{\tau_{M(e)}} \theta T_{es}^{(1)} + \frac{\tau}{\tau_{M(e)}} \theta^2 T_{es}^{(2)} = 0. \quad (18)$$

An analogous equation can be found for the heavy components. Here $\langle c_e \rangle \tau = \lambda_e$ is the mean free path, $eE^* \langle c_e \rangle \tau = \mathscr{E}_e$ field is the energy which an electron acquires from the electric field, $m_e \langle c_e \rangle^2 = \mathscr{E}_e$ kinetic is the average kinetic energy of an electron at temperature T_e, and

$$\frac{eB\tau}{m_e} = \beta_e \cdot \quad (19)$$

is the Hall parameter. There are analogous quantities for the heavy components.

We now assume $\tau = \tau_{M(e)}$, finding

$$\frac{\lambda_e}{L_e} \propto \varepsilon, \quad \frac{L_e}{L_e} \propto 1, \quad \frac{\mathscr{E}_{e\,\text{field}}}{\mathscr{E}_{e\,\text{kinetic}}} \propto \varepsilon, \quad (20)-(26)$$

$$\beta_e \propto 1, \quad \alpha = \frac{n_e}{n_s} \propto 1, \quad \frac{V}{\langle c_e \rangle} \propto \varepsilon, \quad \theta = \varepsilon,$$

and we expand the terms in the two Boltzmann equations in powers of ε; the results are shown in Table 1.

There is no first-order equation for the heavy component; Eq. (27) yields

$$F_e^0 = \left(\frac{m_e}{2\pi k T_e} \right)^{3/2} e^{\frac{-m_e (\vec{W}_e - \vec{V})^2}{2kT_e}}, \quad (32)$$

while Eq. (29) yields

$$F_s^0 = \left(\frac{m_s}{2\pi k T_s} \right)^{3/2} e^{-\frac{m_s (\vec{W}_s - \vec{V})^2}{2kT_s}}. \quad (33)$$

Equation (28) describes the heat flux and electric current due to electron transport; the heavy-component contribution can be neglected in the first approximation (with $\alpha \sim 1$).
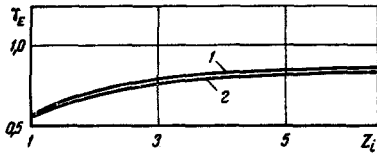
Fig. 1. Dependence of γ_E on the ion charge with $Z_i = 1$. 1) $\gamma_E = 2/(2+\sqrt{2}) = 0.58582$ (according to the present paper); 2) $\gamma_E = 0.5816$ (according to the data of [6]).

<u>Comment.</u> The relaxation times calculated in [2] were based on the distribution functions f_e^0 and f_s^0 written in the form

$$f_e^0 = n_e \left(\frac{m_e}{2\pi k T_e}\right)^{3/2} e^{-\frac{m_e(\vec{W}_e - \vec{V}_e)^2}{2kT_e}}, \quad f_s^0 = n_s \left(\frac{m_s}{2\pi k T_s}\right)^{3/2} e^{-\frac{m_s(\vec{W}_s - \vec{V}_s)^2}{2kT_s}}. \quad (34)$$

However, in the case of a small diffusion Mach number, these functions can be reduced to

$$f_e^0 = n_e \left(\frac{m_e}{2\pi k T_e}\right)^{3/2} e^{-\frac{m_e(\vec{W}_e - \vec{V})^2}{2kT_e}}, \quad f_s^0 = n_s \left(\frac{m_s}{2\pi k T}\right)^{3/2} e^{-\frac{m_s(\vec{W}_s - \vec{V})^2}{2kT_e}}. \quad (35)$$

Equation (28) can be solved by a method similar to the Chapman—Enskog expansion in Sonin polynomials. We will not take up this topic here. We will now use this two-temperature formalism in an attempt to construct a model equation.

III. Construction of a Model Equation. The quantities T^1_{ee} and T^1_{es} in (28) are linear expressions. We know of certain eigenfunctions of the Lorentz operator whose eigenvalues degenerate to

$$\lambda = -\nu(c_e), \quad (36)$$

where ν is the common frequency of direct electron collisions. In the Lorentz case, $\alpha \sim 0$. Analysis of the collision operator shows that this equation takes a form corresponding to that of the BGK formalism [4]. If the interaction is Maxwellian, the collision frequency becomes constant.

Unfortunately, we know nothing about the eigenfunctions of the operator T^1_{ee}. Let us assume that $\vec{c}_e$ is an eigenfunction of T^1_{ee}; then the problem reduces to one of determining the collision frequency. We assume that the collision frequencies are as follows:

(1) for electron—neutral collisions,

$$\nu_{en} = n_n Q_{en} c_e; \quad (37)$$

(2) for electron—ion collisions,

$$\nu_{ei} = n_i Q_{ei} c_e; \quad (38)$$

(3) for electron—electron collisions,

$$\nu_{ee} = n_e Q_{ec} g, \quad (39)$$

where

$$g = |\vec{c}_e - \vec{c}_{ei}|. \quad (40)$$

We denote $\nu_{ei}^{(1)}$ the frequency of Coulomb collisions of electrons corresponding to $Z_i = 1$. According to [5],

$$Q_{ee} = 1/2 Q_{ei}^{(1)}. \quad (41)$$

Replacing g by its average value, $\sqrt{2} c_e$, we find

$$\nu_{ee} = \frac{\sqrt{2}}{2} \nu_{ei}^{(1)}. \quad (42)$$

The corresponding model equation is

$$\vec{c}_e \cdot \frac{\partial \vec{f}_e^0}{\partial r} - \frac{e}{m_e}(\vec{E}^* + \vec{c}_e \times \vec{B}) \cdot \frac{\partial \vec{f}_e^{(1)}}{\partial c_e} + \left[\nu_{en} + \nu_{ei}^{(1)}\left(Z_i^2 + \frac{\sqrt{2}}{2} Z_i\right)\right] f_e^{(1)} = 0. \quad (43)$$

From the solution of this model equation we find the electrical conductivity in the nonequilibrium state:

$$\sigma_{\parallel} = \frac{4\pi e^2}{3kT_e} \int_0^\infty \frac{c_e^4 f_e^0 \left[\nu_{en} + \nu_{ei}^{(1)}\left(Z_i^2 + \frac{\sqrt{2}}{2} Z_i\right)\right] dc_e}{\left[\nu_{en} + \nu_{ei}^{(1)}\left(Z_i^2 + \frac{\sqrt{2}}{2} Z_i\right)\right]^2 + \omega^2}, \quad (44)$$

$$\sigma_{\perp} = \frac{4\pi e^2}{3kT_e} \int_0^\infty \frac{c_e^4 f_e^0 \omega}{\left[\nu_{en} + \nu_{ei}^{(1)}\left(Z_i^2 + \frac{\sqrt{2}}{2} Z_i\right)\right]^2 + \omega^2} dc_e. \quad (45)$$
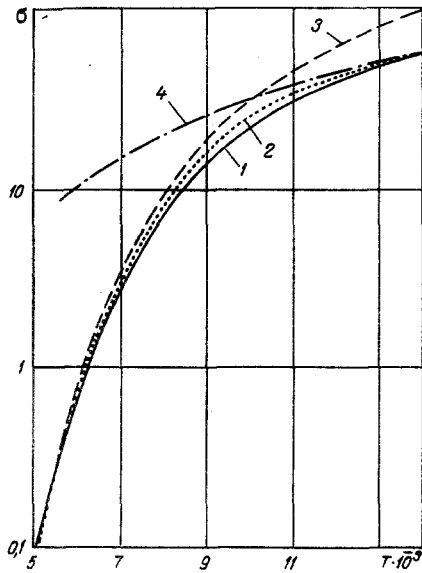
Fig. 2. Temperature dependence of the scalar electrical conductivity (at equilibrium) in argon at atmospheric pressure. 1) Present paper; 2) [7]; 3) Lorentzian gas; 4) [6]. Here T is in Kelvin degrees.

To check this model equation we turn to the available data.

IIIa. Fully Ionized Gas at Equilibrium without a Magnetic Field. Following Spitzer and Harm [6], we introduce the ratio

$$\gamma_E = \frac{\text{non-Lorentzian electrical conductivity}}{\text{Lorentzian electrical conductivity}},$$

for which we find

$$\gamma_E = \frac{Z_i^2}{Z_i^2 + \frac{1}{2}\sqrt{2}\, Z_i}.$$

(46)

The function $\gamma_E = \gamma_E(Z_i)$ is shown in Fig. 1.

IIIb. Partially Ionized Gas at Equilibrium without a Magnetic Field. To check the accuracy of the method in a comparison with the data of Devoto [7] we adopted certain values for the electron—atom cross sections. Figure 2 shows the results for the case of argon at atmospheric pressure in the equilibrium state. We see that the results differ by no more than 15%. Above 6000°K our values are slightly higher than Devoto's, but the discrepancy is not a matter for concern since the Chapman—Enskog method converges extremely slowly at such low temperatures (as Devoto pointed out).

IIIc. Gas in a Magnetic Field. In the case of a fully ionized gas we find the results of [8]; the results for a partially ionized gas are shown in Fig. 3a.

Possible Application of the Method for Heat Conduction. If we assume

$$\left(\frac{m_e c_c^2}{2kT_e} - \frac{5}{2}\right)\vec{c}_e$$

(47)

to be an eigenfunction of the linear operator T^1_{ee}, we run into difficulty in the determination of the corresponding collision frequency. If we assume (this assumption requires justification)

$$\left(\frac{df_e}{dt}\right)^0 + \left[\nu_{en} + \frac{\nu_{ei}}{\gamma_T}\right] f_e^{(1)} = 0,$$

(48)

with $\gamma_T = 0.2252$, as given in [6], we are led to the thermal-conductivity curves in Fig. 3b.

Let us calculate the electronic thermal conductivity in the presence of a magnetic field; in the case of an ionized gas we obviously find the results of [9]. Figures 3c and 4 show curves for argon at equilibrium at atmospheric pressure.

IV. Quasi-Lorentzian Gas. Starting with Eq. (18), and following [10], we carry out an expansion in terms of two parameters:

$$f_e = f_e^0 + \varepsilon\,(f_e^{10} + \alpha f_e^{11}) + \cdots;$$
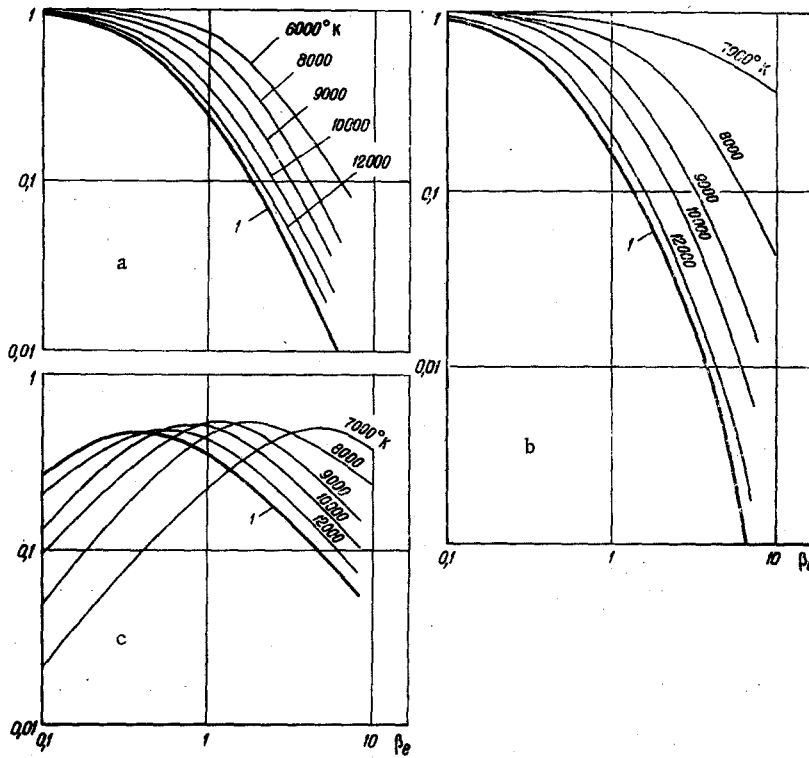
(49)

645

Fig. 3. a) Ratio of the parallel electrical conductivity to the scalar electrical conductivity (curve 1 corresponds to full ionization, according to Sodha and Varshni); b) ratio of the direct thermal conductivity (curve 1 corresponds to full ionization, according to Landshoff); c) ratio of the transverse thermal conductivity to the scalar thermal conductivity (curve 1 corresponds to full ionization, according to Landshoff), for various values of the Hall parameter. These curves are plotted for argon at equilibrium at p = 1 atm.

setting $\varepsilon = \alpha^2$, we find

$$\propto 1 \qquad -\omega_e \, (\vec{\varphi}_e \times \vec{B}) \cdot \frac{\partial \vec{F}_e^0}{\partial \varphi_e} = 0, \tag{50}$$

$$\propto \alpha \qquad T_{ee}^0 = 0, \tag{51}$$

$$\propto \alpha^2 \qquad T_{es}^{(1,0)} + \vec{\varphi}_e \cdot \frac{\partial \vec{F}_e^0}{\partial p} - \vec{\xi} \cdot \frac{\partial \vec{F}_e^0}{\partial \zeta_e} - \beta \, (\vec{\varphi}_e \times \vec{B}) \cdot \frac{\partial \vec{F}_e^{(1,0)}}{\partial \varphi_e} = 0, \tag{52}$$

$$\propto \alpha^3 \qquad T_{ee}^{(1,0)} + T_{es}^{(1,1)} - \beta_e \, (\vec{\varphi}_e \times \vec{B}) \cdot \frac{\partial \vec{F}_e^{(1,1)}}{\partial \varphi_e} = 0 . \tag{53}$$

For the heavy component,

$$\propto \alpha^2 \qquad T_{ss}^0 = 0. \tag{54}$$

Equation (50) contains an isotropic solution, which is found with the help of (51) to be Maxwellian. Solution (52) is identical to the classical Lorentz solution. The next equation gives the non-Lorentzian solution, to which we how turn.

Non-Lorentzian Solutions of the Boltzmann Equation. The conversion to a multicomponent system, achieved by replacing T_{es} by $\sum_{s \neq e} T_{es}$, introduces no new difficulties. With the values of the mass ratio in mind ($m_e/m_s \sim 70{,}000$ for argon), we can identify T_s and T. Setting

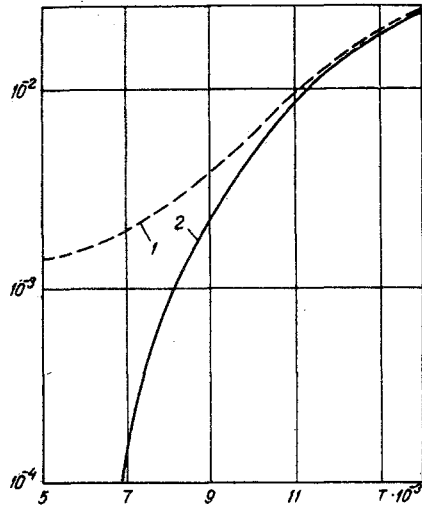$$f_e^{(1,i)} = f_e^* \Phi_e^{(1,i)} \tag{55}$$

Fig. 4. Temperature dependence of the electronic thermal conductivity for argon at equilibrium at p = 1 atm. 1) Total K [7]; 2) K_e (the present paper). Here T is in Kelvin degrees.

where

$$f_e^* = n_e \left(\frac{m_e}{2\pi kT} \right)^{3/2} e^{\frac{-m_e(W_e-V)^2}{2kT}}, \tag{56}$$

we find (i = 0, 1)

$$\sum_{s \neq e} T_{es}^{(1,i)} = \sum_{s \neq e} \int \int f_e^* f_s^0 (\Phi_e^{(1,i)} - \Phi_e^{(1,i)}) \, gb \, db \, d\epsilon \, d_3 c_s = L \, (\Phi_e^{(1,i)}). \tag{57}$$

It is shown in Appendix A that Eqs. (52) and (53) can be reduced to a Fredholm equation with a symmetric kernel, so that we can use the Fredholm theorem to check the integrability. The general solution φ of the corresponding homogeneous equation is a constant (Appendix A). Consequently, a solution exists if and only if the second terms in (52) and (53) are orthogonal with respect to the constant.

The solution $\Phi_e^{(1,0)}$ is the sum of the general solution $\varphi_e^{(1, i)}$ of the corresponding homogeneous equation and the particular solution $\psi_e^{(1, i)}$ of the inhomogeneous equation.

Using conditions (7)-(14), we find the values of the arbitrary parameters:

$$\varphi_e^{(1,0)} = 0, \quad \varphi_e^{(1,1)} = 0. \tag{58}$$

We seek the particular solution $\bar{\psi}^{(1, i)}$ by the classical approximations of direct electron collisions:

$$\vec{g} \simeq \vec{c}_e, \quad c_e^1 \simeq c_e. \tag{59}$$

Accordingly, if $h(c_e)$ is any isotropic function of $\vec{c}_e$, we have

$$L \, [h \, (c_e) \, \vec{c}_e] = - \nu \, (c_e) \vec{c}_e, \tag{60}$$

where $\nu(c_e) = \sum_{s \neq e} \nu_{es}(c_e)$ is the classical frequency of direct electron collisions. The solution of Eq. (52) is the nonequilibrium Lorentz solution, which we write as

$$f_e^{(1,0)} = \frac{-f_e^0 \nu (\vec{G}^* \cdot \vec{c})}{\omega^2 + \nu^2} - \frac{f_e^0 \omega}{\omega^2 + \nu^2} \frac{(\vec{B} \cdot \vec{G}^*)\vec{c}}{B} = f_{e\parallel}^{10} + f_{e\perp}^{10}, \tag{61}$$

where G^* is the classical "driving force."

We assume

$$T_{ee\parallel}^{10} = \tau \, (f_e^0 f_{e\parallel1}^{1,0}) + T(f_{e\parallel}^{1,0} f_{e1}^0), \quad T_{ee\perp}^{10} = \tau \, (f_e^0 f_{e\perp s}^{1,0}) + T(f_{e\perp}^{1,0} f_{e1}^0), \tag{62}$$

where the subscript "1" refers to the "field" electron. Then the non-Lorentzian solution can be written

$$f_e^{(1,1)} = M \, (G^* c_e) + N \left[\frac{\vec{B} \times \vec{G}^*}{B} \cdot \vec{c}_e \right],$$

where M and N are isotropic functions. After determining these functions, we find

647

TABLE 2

	Present paper ($\vec{B}=0$, $T_e = T_g$)	Acc. Spitzer and Harm $Z_i = 1$
$\gamma_E = \dfrac{\sigma_e^{(1,0)} + \sigma_e^{(1,1)}}{\sigma_e^{(1,0)}}$	0,612	0,5816
$\dfrac{\alpha_e^{(1,0)} + \alpha_e^{(1,1)}}{\alpha_e^{(1,0)}}$	0,249	0,2727
$\dfrac{\beta_e^{(1,0)} + \beta_e^{(1,1)}}{\beta_e^{(1,0)}}$	0,476	0,4652
$\gamma_T = \dfrac{k_e^{(1,0)} + k_e^{(1,1)}}{k_e^{(1,0)}}$	0,2765	0,2252

$$f_e^{1,1} = \frac{-v T_{ee}^{1,0}}{v^2 + \omega^2} + \frac{\omega}{v^2 + \omega^2}\left[\frac{\vec{G^*}\cdot\vec{c}_e}{(\vec{B}\times\vec{G^*})\cdot\vec{c}_e/B} T_{ee\perp}^{1,0} + \frac{(\vec{B}\times\vec{G^*})\cdot\vec{c}_e/B}{(\vec{G^*}\cdot\vec{c}_e)} T_{ee}^{1,0} \right]. \tag{63}$$

From this solution we can determine the analytic expressions for the nonequilibrium non-Lorentzian transport coefficients. If there is no magnetic field, $\vec{B} = 0$, setting

$$n_e^2 I^* (A) = T(f_e^0 f_{e1}^* A_1) + T(f_e^* A f_{e1}^0)$$

and

$$\Omega^{m,n} = \frac{1}{n_e^2}\iint \frac{c_e^m \vec{c}_e}{v} I^*\left(\frac{c_e^n \vec{c}_e}{v} \right) d_3 c_e = \Omega^{m,n},$$

we find

$$\sigma_e^{(1,1)} = -\frac{n_e^2 e^2}{3kT_e}\Omega^{0,0}, \tag{64}$$

$$\alpha_e^{(1,1)} = -\frac{n_e^2 m_e}{6}\left[\frac{m_e}{2kT_e}\Omega^{0,2} - \frac{5}{2}\Omega^{0,0} \right], \tag{65}$$

$$\beta_e^{(1,1)} = -\frac{n_e^2 m_e}{6}\Omega^{2,0}, \tag{66}$$

$$K_e^{(1,1)} = -\frac{m_e n_e^2}{6 T_e}\left[\frac{m_e}{2kT_e}\Omega^{2,2} - \frac{5}{2}\Omega^{2,0} \right]. \tag{67}$$

In this nonequilibrium case we must introduce a restriction which follows from the nonequilibrium Debye length (Appendix B):

$$d' = \left(\frac{2kk_0 T_e T}{n_e e^2 (T_e + T)} \right)^{1/2}. \tag{68}$$

Integrals like those in Eq. (63) are not amenable to numerical calculations; the calculations can probably be carried out by the Monte Carlo method. Calculations were carried out in [11] for the case in which v is some polynomial expansion in the electron velocity c_e. The present data refer to the case of a purely Coulomb interaction, so that they can be compared with the data of [6] (Table 2). The discrepancy here is to be expected, since a fully ionized gas is not a quasi-Lorentzian gas. In the case of a Maxwell interaction ($v = \text{const}$), the quantity $\sigma_e^{1,1}$ vanishes.

V. General Conclusions. Working from the results of [2], which dealt with the relaxation times in an inhomogeneous gas with a small Mach number, we have developed a small parameter method for solving the Boltzmann equation. This method can be used if the electron and gas temperatures are different. This method is an extension of the Chapman—Enskog method for the small parameter $\varepsilon = (m_e/m_s)^{1/2}$. As was shown in [2], the term describing the collisions between electrons and other species has two related characteristic times, $\tau_{E(e)}$ and $\tau_{M(e)}$, which are very different in magnitude and refer to energy and momentum exchange, respectively.

The unknown distribution functions are expanded in series; the zeroth-order terms take into account the difference between the relaxation temperature, while the other terms take into account momentum

exchange. A method of successive approximations has been worked out in which the zeroth-order terms $f_e^0 (n_e, T_e, \vec{V})$ and f_s^0 yield the Maxwell solution. This solution corresponds to the relaxation times given in [2].

The nonequilibrium transport coefficients are calculated by the Chapman—Enskog procedure, except that the temperatures T_e and T_s are different. A magnetic field introduces no new difficulties.

We attempted to find analytic results by two methods; first, we transformed the linearized equation into a generalized BGK equation on the basis of a crude estimate of the eigenvalue. This approach yielded numerical results in good agreement with results from the literature [6, 9].

Second, following Eucken, we introduced a small parameter, the degree of ionization $\alpha = n_e/n_s$, so that this system of equations is valid for the case of a quasi-Lorentzian gas. The zeroth-order solution is still Maxwellian, while the first-order solution corresponds to the Lorentz solution (with $T_e \neq T_s$), and the third-order solution gives the non-Lorentzian correction to the transport coefficients, which are calculated analytically as Chapman—Cowling integrals. The numerical results found for a fully ionized gas agree well with those obtained in [6], although in this particular case the mixture is quite different from the Lorentz model.

APPENDIX A

We consider the linear operator

$$L(\varphi) = \iint f_e^* f_s^0 (\Phi - \Phi') k_{cs} d_2 k d_3 c_s. \tag{A.1}$$

We can show that this is a Fredholm operator with a symmetric kernel. First, we write

$$\iint f_e^* f_s^0 \Phi k_{cs} d_2 k d_3 c_s = k_0 (\vec{c_e}) \Phi. \tag{A.2}$$

Let us examine the integral

$$I = \iint f_e^* f_s^0 \Phi' k_{es} d_2 k d_3 c_s. \tag{A.3}$$

We assume an elastic collision in velocity space (Fig. A.1):

$$\vec{g} = \vec{c_s} - \vec{c_c}. \tag{A.4}$$

Following Chapman and Cowling [13], we introduce the new variables

$$\vec{k} = g\vec{\mathrm{k}}, \quad \vec{g} = g\vec{n}. \tag{A.5}$$

We find

$$d_3 k = g^2 dg d_2 \mathrm{k}, \tag{A.6}$$

$$d_3 g = g^2 dg d_2 n, \tag{A.7}$$

Here $\vec{c_e}$ must be treated as a fixed quantity; then

$$d_3 c_s = d_3 g \tag{A.8}$$

and thus

$$I = \iint f_e^* f_s^0 \Phi' k_{es} (g, \psi) d_2 n d_3 k. \tag{A.9}$$

Introducing the new variable

$$\vec{K} = (\vec{g} \cdot \vec{\mathrm{k}})\mathrm{k} = g \cos \psi k = \cos \psi \vec{k}, \tag{A.10}$$

we have

$$d_3 \mathrm{k} = \cos^3 \psi d_3 k. \tag{A.11}$$

On the other hand,

$$\vec{c_c} = \vec{c_e} + \frac{2m_s}{m_e + m_s} (\vec{g} \cdot \vec{\mathrm{k}})\vec{\mathrm{k}} \tag{A.12}$$

and

$$I = \iint f_e^* f_s^0 \Phi' k_{es} \left(\frac{K}{\cos \psi}, \quad \psi \right) \frac{d_2 n d_3 c_e'}{\cos^3 \psi}, \tag{A.13}$$
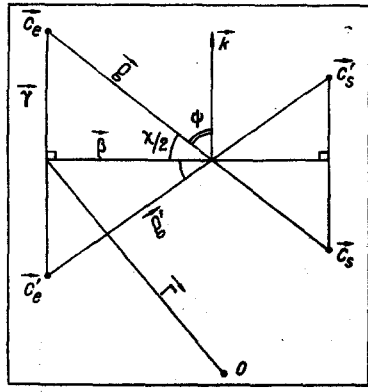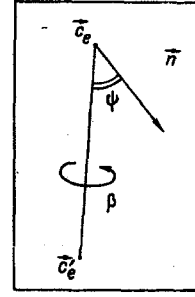
Fig. A.1   Fig. A.2

where $K = |\vec{c_e'} - \vec{c_e}|/2m_s$ is symmetric with respect to $\vec{c_e}$ and $\vec{c_e'}$. Then the kernel is

$$\Lambda = \int f_e^* f_s^0 k_{es}\left(\frac{K}{\cos\psi}, \psi\right)\frac{1}{\cos^3\psi}\, d_2 n. \tag{A.14}$$

In this integral $\vec{c_e}$ and $\vec{c_e'}$ must be treated as fixed. We consider the plane formed by the two vectors $(\vec{c_e} - \vec{c_e'})$ and $\vec{n}$; it is governed by the angle β (Fig. A.2). Since

$$d_2 n = \sin\psi\, d\psi\, d\beta, \tag{A.15}$$

then

$$\Lambda = \int \frac{1}{\cos^3\psi} k_{es}\left(\frac{K}{\cos\psi}, \psi\right)\sin\psi\, d\psi \int_0^{2\pi} f_e^* f_s^0 d\beta. \tag{A.16}$$

For Λ to be symmetric, the quantity $\int_0^{2\pi} f_e^* f_s^0 d\beta$ must be symmetric with respect to $(\vec{c_e}, \vec{c_e'})$. Following Kogan [12], we write

$$f_e^* f_s^0 = (f_e^* f_e^{*\prime})^{1/2}(f_s^0 f_s^{\prime 0})^{1/2}. \tag{A.17}$$

We assume

$$\vec{\Gamma} = \frac{\vec{c_e} + \vec{c_e'}}{2}, \tag{A.18}$$

$$\vec{\gamma} = \frac{\vec{c_e} - \vec{c_e'}}{2}. \tag{A.19}$$

If $\vec{\beta}$ is the unit vector shown in Fig. A.1, then we have

$$\int_0^{2\pi} e^{-\frac{2(m_e + m_s)}{kT}\gamma\,\mathrm{tg}\psi(\vec{\Gamma}\cdot\vec{\beta})}\, d\beta. \tag{A.20}$$

We write

$$\vec{\Gamma}\cdot\vec{\beta} = \Gamma\cos\beta\sin\theta, \tag{A.21}$$

where $\cos\theta = c_e^2 - c_e^{\prime 2}/\Gamma\gamma$ (asymmetric). If we write the exponential function in series form, we see that only odd powers of $\sin\theta$ make nonvanishing contributions.

Accordingly, the integral and the kernel are symmetric.

In conclusion we can write

$$\iint f_e^* f_s^0(\Phi - \Phi')k_{es}d_2 k d_3 c_s = K_0(\vec{c_e})\Phi + \int K_1(\vec{c_e}, \vec{c_e'})\Phi' d_3 c_e'. \tag{A.22}$$

This equation and Eqs. (52) and (53) of the text proper are Fredholm equations with symmetric kernels.

Considering the homogeneous equation

$$\iint f_e^* f_s^0(\varphi - \varphi')k_{es}d_2 k d_3 c_s = 0, \tag{A.23}$$

we find

$$\int \varphi \int \int f_e^* \, f_s^0 \, (\varphi - \varphi') \, k_{es} d_2 k d_3 c_s d_3 c_e$$

$$= \frac{1}{2} \int \int \int f_e^* \, f_s^0 (\varphi - \varphi')^2 k_{es} d_2 k d_3 c_s d_3 c_e. \tag{A.24}$$

Solution (A.21) satisfies the condition $\varphi = \varphi'$. Then for binary electron collisions the only solution is $\varphi = \text{const}$. Now we can easily verify that the Fredholm integrability condition holds for Eqs. (52) and (53).

APPENDIX B

We donote by Φ the electric potential of the electrons and ions; then the Poisson equation is

$$\Delta\Phi + \frac{\rho_e + \rho_i}{4\pi k_0} = 0. \tag{B.1}$$

We replace the distribution functions by the following Maxwell functions:

$$f_e^0 = n_{e0} \left(\frac{m_e}{2\pi k T_e} \right)^{3/2} e^{\left[-\frac{m_e c_e^2}{2kT_e} + \frac{e\Phi}{kT_e} \right]}, \tag{B.2}$$

$$f_i^0 = n_{i0} \left(\frac{m_i}{2\pi k T_i} \right)^{3/2} e^{\left[-\frac{m_i c_i^2}{2kT_i} - \frac{Z_i e\Phi}{kT_i} \right]}. \tag{B.3}$$

Then we find

$$n_e = n_{e0} e^{e\Phi/kT_e}, \tag{B.4}$$

$$n_i = n_{i0} e^{-Z_i e\Phi/kT_i}. \tag{B.5}$$

Substituting these equations into the Poisson equation, written in terms of spherical coordinates, and linearizing the exponential functions, we find

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) = \frac{e^2 n_{e0}}{k k_0} \Phi \left(\frac{Z_i}{T_i} + \frac{1}{T_e} \right), \tag{B.6}$$

whose solution is

$$\Phi = \Phi_s e^{-\sqrt{1 + Z_i}\, r/d'}. \tag{B.7}$$

Hence the Debye length in the nonequilibrium case is

$$d' = \left(\frac{k k_0 (1 + Z_i) T_e T_i}{n_{e0} e^2 (Z_i T_e + T_i)} \right)^{1/2}. \tag{B.8}$$

NOTATION

f_e, f_s	are the distribution functions;
F_e, F_s	are the dimensionless distribution functions;
$\vec{W}_e$, $\vec{W}_s$	are the absolute velocities;
$\vec{r} = (x, y, z)$	is the radius vector in coordinate space;
$\vec{\partial}/\partial r$	is the gradient operator in coordinate space;
$\vec{\partial}/\partial n$	is the gradient operator in velocity space;
m_e, m_s	are the masses;
n_i, n_n, n_e, n_s	are the densities;
$\vec{V}$	is the macroscopic velocity of mixture;
$\vec{V}_e$, $\vec{V}_s$	are the macroscopic velocities for the components of the mixture;
T	is the mixture temperature;
T_e, T_s	are the component temperatures;
$\vec{c}_e = \vec{W}_e - \vec{V}$	are the diffusion velocities;
$\langle c_e \rangle$, $\langle c_s \rangle$	are the average diffusion velocities;
e	is the electron charge;
eZ_i	is the ion charge;
$\vec{E}$, $\vec{B}$	are the electric and magnetic fields;
T_{ee}, T_{ss}	are the integrals for elastic collisions between particles of the same species;

T_{es}, T_{se}	are the integrals for elastic collisions between particles of different species;
τ_{ee}, τ_{ss}	are the self-relaxation times;
$\tau_{E(e)}$, $\tau_{E(s)}$	are the thermal relaxation times;
$\tau_{M(e)}$, $\tau_{M(s)}$	are the relaxation times for momentum exchange;
$\varepsilon = (m_e/m_s)^{1/2}$	is the small parameter;
$\alpha = n_e/n_s$	is the degree of ionization;
$\vec{\varphi}_e$, $\vec{\varphi}_s$	are the dimensionless velocities;
$\vec{\xi}$, $\vec{v}$, $\vec{B}$, $\vec{p}$	are the dimensionless quantities;
τ	is the characteristic time;
λ_e	is the electron mean free path;
$\mathscr{E}$ e field	is the energy acquired by an electron from the field;
$\mathscr{E}$ e kinetic	is the average kinetic energy of an electron at temperature T_e;
ω	is the electron gyrofrequency;
β_e	is the Hall parameter;
ν_{es}	is the frequency of collisions between electrons and particles of species s;
$\nu = \displaystyle\sum_{s \neq e} \nu_{es}$	is the total frequency of collisions between electrons and heavy particles;
Q_{en}, Q_{ei}, Q_{ee}	are the cross sections for electron—atom, electron—ion, and electron—electron interactions;
$\vec{g} = \vec{c}_e - \vec{c}_{ei}$	is the relative velocity;
$\sigma_\parallel$	is the parallel (or "ordinary") electrical conductivity;
$\sigma_\perp$	is the normal (or transverse) electrical conductivity;

$$\gamma_E = \frac{\text{non-Lorentzian electrical conductivity}}{\text{Lorentzian electrical conductivity}} \; ;$$

$$\gamma_T = \frac{\text{non-Lorentzian thermal conductivity}}{\text{Lorentzian thermal conductivity}} \; ;$$

σ_e, α_e, β_e, K_e	are the transport coefficients;
d'	is the nonequilibrium Debye length;
χ	is the scattering angle.

LITERATURE CITED

1. H. Grad, "Theory of rarefied gases," in: Rarefied Gas Dynamics (ed. F. M. Devienne), Pergamon, New York—London (1960).
2. T. F. Morse, Phys. Fluids, 6, No. 10 (1963).
3. E. A. Johnson, Phys. Fluids, 16, No. 1 (1973).
4. P. L. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev., 94, No. 3, 511 (1954).
5. G. W. Sutton and A. Sherman, Engineering Magnetohydrodynamics, McGraw—Hill, New York (1965).
6. L. Spitzer, Jr., and R. Harm, Phys. Rev., 89, 977 (1953).
7. R. S. Devoto, Phys. Fluids, 10, No. 2 (1967).
8. M. S. Sodha and Y. P. Varshni, Phys. Rev., 111, 1203-1205 (1958); "Dependence of electron mobility on magnetic field in a fully ionized gas," Phys. Rev., 114, 946-947 (1959).
9. R. Landshoff, Phys. Rev., 76, 904 (1949).
10. A. Eucken, Physik Z., 14, 324 (1913); M. N. Kogan, Rarefied Gas Dynamics, Plenum, New York (1969), p. 216.
11. J.-P. Petit, Journal de Mécanique, 11, No. 2, 233 (1972).
12. M. N. Kogan, Rarefied Gas Dynamics (Kinetic Theory) [in Russian], Nauka, Moscow (1967).
13. S. Chapman and T. G. Cowling, Mathematical Theory of Nonuniform Gases, Cambridge Univ. Press (1953).
14. C. Cercignani, Mathematical Methods in Kinetic Theory, Macmillan, New York (1969).
15. T. F. Morse, Phys. Fluds, 7, No. 2, 159-169 (1964).
16. R. M. Chmielski and J. H. Ferziger, Phys. Fluids, 10, No. 2 (1967).